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# INTERFACIAL DEBONDING OF A CIRCULAR INHOMOGENEITY IN PIEZOELECTRIC MATERIALS

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**Abstract**—A generalized and mathematically rigorous model is developed to treat the partially-debonded circular inhomogeneity problem in piezoelectric materials under antiplane shear and in plane electric field using the complex variable method. The principle of analytical continuation and the complex series expansion method were employed to reduce the formulations into Riemann–Hilbert problems. This enabled the explicit determination of the complex potentials inside the inhomogeneity and the matrix. The resulting closed form expressions were then used to obtain the energy release rate for several interesting cases involving partial-debonding at the inhomogeneity-matrix interface. © 1997 Elsevier Science Ltd.

## 1. INTRODUCTION

In view of their potential use in large space and aircraft structures, satellites and other applications, smart materials and adaptive structures have gained significant attention from the scientific community. For example, piezoelectric ceramics have attracted attention as sensors for monitoring and as actuators for controlling the behavior of advanced aerospace structures. Piezoelectric composites in the form of electromechanical transducers have also been used in underwater hydrophones and medical images devices.

Despite the extensive efforts currently being devoted to the development of piezoelectric composite materials, the theoretical understanding of these materials is quite limited in comparison with the uncoupled response of traditional composites. This may be partly due to the complexities arising from the electromechanical coupling and partly to the scarcity of experimental data.

Piezoelectric composite materials are generally characterized by heterogeneity, anisotropy, load sharing and interfaces. Indeed, increasing evidence suggests that the presence of inhomogeneities in the form of second phase particles, precipitates, fibre reinforcement, voids and microcracks often govern the mechanical and electric behaviour and the overall failure mechanism of these solids. For example, the thermal expansion mismatch between the fibres and the host matrix of most composites combined with the large temperature change during fabrication and cooldown results in high tensile residual stresses which lead to matrix microcracking and/or fibre debonding. Accordingly, an accurate assessment of the influence of these inhomogeneities upon the mechanical and electric behaviour and the integrity of these composites would be of great value to engineering designers and material scientists.

Indeed, a number of contributions concerning the inhomogeneity problem in piezoelectric materials have appeared in the literature. These include the work of Deeg (1980), Wang (1992), Benveniste (1992) and Dunn and Taya (1993), among others, on the fully-bonded inhomogeneity problem. Efforts have also been devoted to the straight crack problem in this important class of materials, see, for example, the work of McMeeking (1989), Pak (1990), Sosa and Pak (1990), Suo *et al.* (1992), Yang and Suo (1994), Park and Sun (1995), Zhang and Tong (1996).

In this article, however, a generalized and mathematically rigorous model is developed to treat the partially-debonded circular inhomogeneity problem in piezoelectric materials

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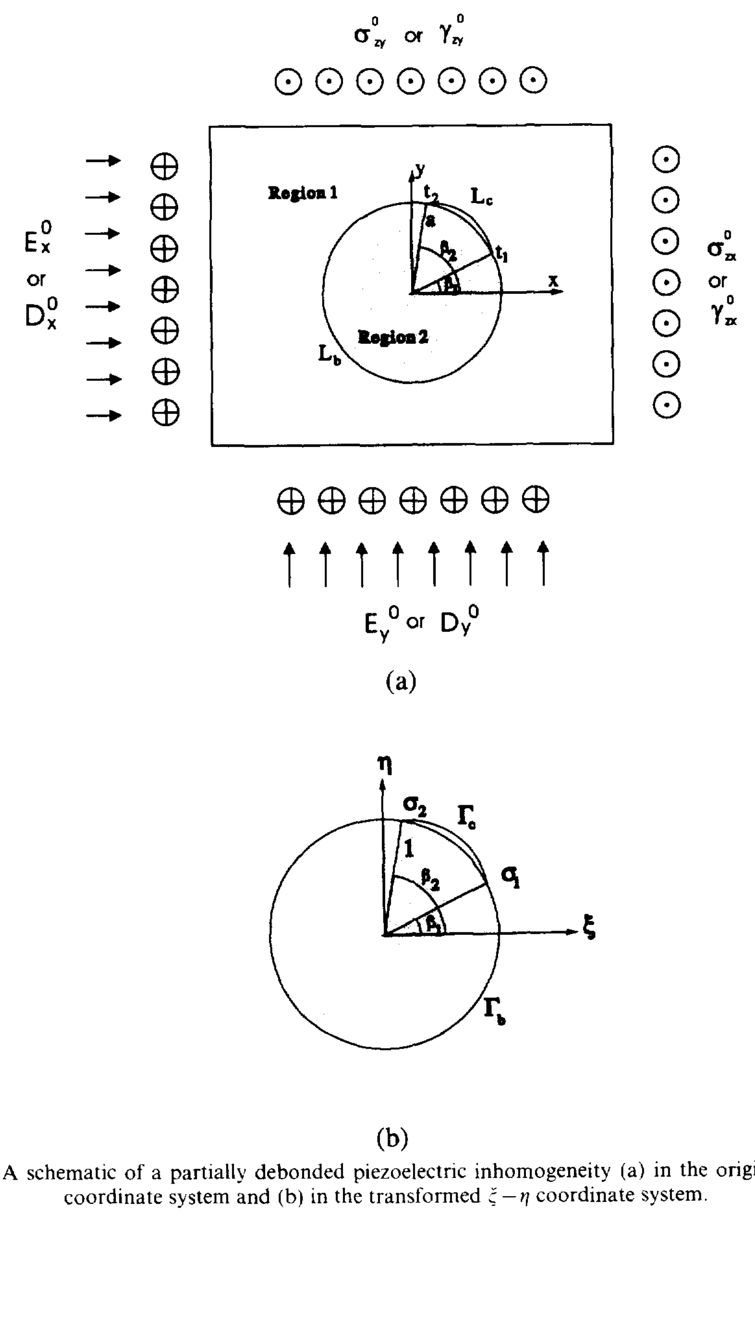
under antiplane shear and inplane electric field using the complex variable method. The principle of analytical continuation and the complex series expansion method are employed to reduce the formulations into two Riemann–Hilbert problems. This enables the explicit determination of the complex potentials of the problem and the energy release rate at the tips of the resulting curvilinear crack.

This article is divided into five sections. Following this brief introduction, Section 2 outlines the basic equations needed to formulate the problem. In Section 3, we provide the approach adopted in reaching a solution. A number of interesting cases are examined in Section 4 and finally we conclude the paper.

## 2. PROBLEM STATEMENT AND FORMULATION

Consider a partially debonded circular piezoelectric inhomogeneity with a radius $a$ in an infinite piezoelectric matrix subject to an inplane ($x-y$ or $r-\theta$ plane) electric field as well as an antiplane shear at infinity, as shown in Fig. 1(a). The inhomogeneity and the matrix have different elastic, electric and piezoelectric properties and both are assumed to be transversely isotropic with respect to the longitudinal direction ($z$ direction).




(b)

Fig. 1. A schematic of a partially debonded piezoelectric inhomogeneity (a) in the original $x-y$ coordinate system and (b) in the transformed $\xi-\eta$ coordinate system.

The respective regions occupied by the matrix and the inhomogeneity will be referred to as regions 1 and 2 and the quantities associated with the matrix and the inhomogeneity will be denoted by the corresponding superscripts or subscripts. The interface between regions 1 and 2 is denoted by $L(=L_c+L_b)$, where $L_c$ represents an insulated and a traction-free interfacial curvilinear crack and $L_b$ is the remaining part of $L$ along which the inhomogeneity and the matrix are perfectly bonded.

For this problem, the out-of-plane displacement $w$ and the electric potential $\phi$ are only functions of the variables $x$ and $y$, such that

$$w = w(x,y) \quad \phi = \phi(x,y). \tag{1}$$

The equilibrium equations for the stresses and the electric displacements are

$$\begin{aligned} &\frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{zy}}{\partial y}=0 \\ &\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}=0 \end{aligned} \tag{2}$$

where $\sigma_{zx}$ and $\sigma_{zy}$ are the shear stresses, while $D_x$ and $D_y$ are the electric displacements.

For linear piezoelectric materials, the constitutive relations can be written as

$$\begin{aligned} \sigma_{zx} &= G_L\gamma_{zx}-e_{15}E_x \\ \sigma_{zy} &= G_L\gamma_{zy}-e_{15}E_y \\ D_x &= e_{15}\gamma_{zx}+k_{11}E_x \\ D_y &= e_{15}\gamma_{zy}+k_{11}E_y \end{aligned} \tag{3}$$

where $\gamma_{zx}$ and $\gamma_{zy}$ are the shear strains, $E_x$ and $E_y$ are the electric fields, $G_L$ is the longitudinal shear modulus, $e_{15}$ denotes the piezoelectric modulus and $k_{11}$ represents the dielectric modulus.

The shear strains $\gamma_{zx}$ and $\gamma_{zy}$ and the electric fields $E_x$ and $E_y$ are related to the displacement $w$ and the electric potential $\phi$ by the following relations:

$$\gamma_{zx}=\frac{\partial w}{\partial x} \quad \gamma_{zy}=\frac{\partial w}{\partial y} \quad E_x=-\frac{\partial \phi}{\partial x} \quad E_y=-\frac{\partial \phi}{\partial y}. \tag{4}$$

Substituting (3) and (4) into (2), we obtain the following governing equations:

$$\nabla^2 w=0 \quad \nabla^2\phi=0 \tag{5}$$

where $\nabla^2$ is the Laplacian operator.

The boundary conditions at the interface between the inhomogeneity and the matrix can be expressed as

$$\sigma_{zr1}=\sigma_{zr2}=0 \quad D_{r1}=D_{r2}=0 \quad \text{on } L_c \tag{6}$$

$$\sigma_{zr1}=\sigma_{zr2} \quad w_1=w_2 \quad D_{r1}=D_{r2} \quad \phi_1=\phi_2 \quad \text{on } L_b \tag{7}$$

which describe the electrically impermeable and traction-free conditions along the curvilinear crack. The condition that the normal component of the electric displacement can be taken as zero at the crack faces involves two assumptions; namely: (1) that no free charge resides on the crack face and (2) that the electric displacement within the crack is negligible. For a homogeneous material with a slit crack, the model of electrically impermeable cracks is appropriate when the aspect ratio of the flaw thickness to its length is much larger than the ratio of the dielectric constant of the flaw to that of the material (McMeeking, 1989; Zhang and Tong, 1996). For the current problem where there are three electric media (flaw and two bonded materials), the assumption of electrically impermeable cracks should be examined further. This is currently being persued by the authors.

For simplicity, a new dimensionless coordinate system $(\xi, \eta)$ is introduced with $\xi = x/a$ and $\eta = y/a$, as shown in Fig. 1(b). In this coordinate system, regions 1 and 2 are transformed, respectively, into the exterior and the interior regions of a unit circle $\Gamma$. In this case, $\Gamma$ consists of two parts $\Gamma_c$ and $\Gamma_b$ which correspond to $L_c$ and $L_b$ in the original coordinate system $(x, y)$. The points $\sigma_1 = e^{i\beta_1}$ and $\sigma_2 = e^{i\beta_2}$ in the new coordinate system correspond to the tips of the interfacial crack, $t_1 = ae^{i\beta_1}$ and $t_2 = ae^{i\beta_2}$ in the original coordinate system.

Equation (5) indicates that $w$ and $\phi$ are harmonic functions which can be taken as the real part of some analytic functions of the complex variable $Z = \xi + i\eta$, such that

$$w = \frac{1}{2G_L}(\Psi(Z) + \overline{\Psi(Z)})$$

$$\phi = \frac{1}{2k_{11}}(\Phi(Z) + \overline{\Phi(Z)}) \tag{8}$$

where $\Psi$ and $\Phi$ represent the analytic functions and the overbar denotes the complex conjugate. Accordingly, the stress and the electric displacement can be expressed as

$$\sigma_{zx} - i\sigma_{zy} = \frac{1}{a}\left[\Psi'(Z) + \frac{e_{15}}{k_{11}}\Phi'(Z)\right]$$

$$D_x - iD_y = \frac{1}{a}\left[\frac{e_{15}}{G_L}\Psi'(Z) - \Phi'(Z)\right] \tag{9}$$

where the prime denotes the derivative with respect to the argument.

In terms of polar coordinates $r$ and $\theta$, the above expressions become

$$w_{,\theta} = \frac{i}{2G_L}[Z\Psi'(Z) - \overline{Z\Psi'(Z)}]$$

$$\phi_{,\theta} = \frac{i}{2k_{11}}[Z\Phi'(Z) - \overline{Z\Phi'(Z)}] \tag{10}$$

and

$$\sigma_{zr} - i\sigma_{z\theta} = \frac{Z}{a|Z|}\left(\Psi'(Z) + \frac{e_{15}}{k_{11}}\Phi'(Z)\right)$$

$$D_r - iD_\theta = \frac{Z}{a|Z|}\left(\frac{e_{15}}{G_L}\Psi'(Z) - \Phi'(Z)\right). \tag{11}$$

Substituting (10) and (11) into (6) and (7) leads to

$$\sigma\Psi_1'(\sigma) + \overline{\sigma\Psi_1'(\sigma)} + \frac{e_{15}^1}{k_{11}^1}[\sigma\Phi_1'(\sigma) + \overline{\sigma\Phi_1'(\sigma)}] = 0 \quad \sigma \in \Gamma_c \tag{12}$$

$$\sigma\Psi_2'(\sigma) + \overline{\sigma\Psi_2'(\sigma)} + \frac{e_{15}^2}{k_{11}^2}[\sigma\Phi_2'(\sigma) + \overline{\sigma\Phi_2'(\sigma)}] = 0 \quad \sigma \in \Gamma_c \tag{13}$$

$$\frac{e_{15}^1}{G_L^1}[\sigma\Psi_1'(\sigma)+\overline{\sigma\Psi_1'(\sigma)}]-[\sigma\Phi_1'(\sigma)+\overline{\sigma\Phi_1'(\sigma)}]=0 \quad \sigma\in\Gamma_c \tag{14}$$

$$\frac{e_{15}^2}{G_L^2}[\sigma\Psi_2'(\sigma)+\overline{\sigma\Psi_2'(\sigma)}]-[\sigma\Phi_2'(\sigma)+\overline{\sigma\Phi_2'(\sigma)}]=0 \quad \sigma\in\Gamma_c \tag{15}$$

$$\sigma\Psi_1'(\sigma)+\overline{\sigma\Psi_1'(\sigma)}+\frac{e_{15}^1}{k_{11}^1}[\sigma\Phi_1'(\sigma)+\overline{\sigma\Phi_1'(\sigma)}]$$
$$=\sigma\Psi_2'(\sigma)+\overline{\sigma\Psi_2'(\sigma)}+\frac{e_{15}^2}{k_{11}^2}[\sigma\Phi_2'(\sigma)+\overline{\sigma\Phi_2'(\sigma)}] \quad \sigma\in\Gamma_b \tag{16}$$

$$\frac{e_{15}^1}{G_L^1}[\sigma\Psi_1'(\sigma)+\overline{\sigma\Psi_1'(\sigma)}]-[\sigma\Psi_1'(\sigma)+\overline{\sigma\Phi_1'(\sigma)}]$$
$$=\frac{e_{15}^2}{G_L^2}[\sigma\Psi_2'(\sigma)+\overline{\sigma\Psi_2'(\sigma)}]-[\sigma\Phi_2'(\sigma)+\overline{\sigma\Phi_2'(\sigma)}] \quad \sigma\in\Gamma_b \tag{17}$$

$$\mu_1[\sigma\Psi_1'(\sigma)-\overline{\sigma\Psi_1'(\sigma)}]=\sigma\Psi_2'(\sigma)-\overline{\sigma\Psi_2'(\sigma)} \quad \sigma\in\Gamma_b \tag{18}$$

$$\mu_2[\sigma\Phi_1'(\sigma)-\overline{\sigma\Phi_1'(\sigma)}]=\sigma\Phi_2'(\sigma)-\overline{\sigma\Phi_2'(\sigma)} \quad \sigma\in\Gamma_b \tag{19}$$

where $\mu_1 = G_L^2/G_L^1$, $\mu_2 = k_{11}^2/k_{11}^1$.

The main task now is to obtain the complex potentials which satisfy the continuity conditions given by eqns (12)–(19). In the next section, a general solution is presented for the problem in which the complex potentials are derived explicitly for both the inhomogeneity and the matrix.

## 3. SOLUTION

For convenience, let us introduce the following new functions

$$u_1(Z)=Z\Psi_1'(Z) \quad |Z|>1 \tag{20}$$

$$u_2(Z)=Z\Psi_2'(Z) \quad 0<|Z|<1 \tag{21}$$

$$v_1(Z)=Z\Phi_1'(Z) \quad |Z|>1 \tag{22}$$

$$v_2(Z)=Z\Phi_2'(Z) \quad 0<|Z|<1 \tag{23}$$

within the matrix or the inhomogeneity. Using the principle of analytical continuation, the above functions are extended into their complementary regions such that

$$u_1(Z)=-\overline{u_1\left(\frac{1}{\bar{Z}}\right)}=-\frac{1}{Z}\overline{\Psi_1'}\left(\frac{1}{Z}\right) \quad 0<|Z|<1 \tag{24}$$

$$u_2(Z)=-\overline{u_2\left(\frac{1}{\bar{Z}}\right)}=-\frac{1}{Z}\overline{\Psi_2'}\left(\frac{1}{Z}\right) \quad |Z|>1 \tag{25}$$

$$v_1(Z)=-\overline{v_1\left(\frac{1}{\bar{Z}}\right)}=-\frac{1}{Z}\overline{\Phi_1'}\left(\frac{1}{Z}\right) \quad 0<|Z|<1 \tag{26}$$

$$v_2(Z) = -\overline{v_2\left(\frac{1}{\bar{Z}}\right)} = -\frac{1}{Z}\overline{\Phi_2'}\left(\frac{1}{Z}\right) \quad |Z| > 1. \tag{27}$$

Hence, the boundary conditions (12)–(19) can be written as

$$[u_1^+(\sigma) - u_1^-(\sigma)] + \frac{e_{15}^1}{k_{11}^1}[v_1^+(\sigma) - v_1^-(\sigma)] = 0 \quad \sigma \in \Gamma_c \tag{28}$$

$$[u_2^+(\sigma) - u_2^-(\sigma)] + \frac{e_{15}^2}{k_{11}^2}[v_2^+(\sigma) - v_2^-(\sigma)] = 0 \quad \sigma \in \Gamma_c \tag{29}$$

$$\frac{e_{15}^1}{G_L^1}[u_1^+(\sigma) - u_1^-(\sigma)] - [v_1^+(\sigma) - v_1^-(\sigma)] = 0 \quad \sigma \in \Gamma_c \tag{30}$$

$$\frac{e_{15}^2}{G_L^2}[u_2^+(\sigma) - u_2^-(\sigma)] - [v_2^+(\sigma) - v_2^-(\sigma)] = 0 \quad \sigma \in \Gamma_c \tag{31}$$

$$[u_1^+(\sigma) - u_1^-(\sigma)] + \frac{e_{15}^1}{k_{11}^1}[v_1^+(\sigma) - v_1^-(\sigma)] = -[u_2^+(\sigma) - u_2^-(\sigma)] - \frac{e_{15}^2}{k_{11}^2}[v_2^+(\sigma) - v_2^-(\sigma)] \quad \sigma \in \Gamma_b \tag{32}$$

$$\frac{e_{15}^1}{G_L^1}[u_1^+(\sigma) - u_1^-(\sigma)] - [v_1^+(\sigma) - v_1^-(\sigma)] = -\frac{e_{15}^2}{G_L^2}[u_2^+(\sigma) - u_2^-(\sigma)] + [v_2^+(\sigma) - v_2^-(\sigma)] \quad \sigma \in \Gamma_b \tag{33}$$

$$\mu_1[u_1^+(\sigma) + u_1^-(\sigma)] - [u_2^+(\sigma) + u_2^-(\sigma)] = 0 \quad \sigma \in \Gamma_b \tag{34}$$

$$\mu_2[v_1^+(\sigma) + v_1^-(\sigma)] - [v_2^+(\sigma) + v_2^-(\sigma)] = 0 \quad \sigma \in \Gamma_b \tag{35}$$

where the superscripts + and − are used to denote the limit of a function as $Z$ tends to $\sigma$ from $|Z| > 1$ and $|Z| < 1$, respectively.

From (28)–(33), it is easy to show that

$$u_1^+(\sigma) + u_2^+(\sigma) + \frac{e_{15}^1}{k_{11}^1}v_1^+(\sigma) + \frac{e_{15}^2}{k_{11}^2}v_2^+(\sigma) = u_1^-(\sigma) + u_2^-(\sigma) + \frac{e_{15}^1}{k_{11}^1}v_1^-(\sigma) + \frac{e_{15}^2}{k_{11}^2}v_2^-(\sigma) \quad \sigma \in \Gamma = \Gamma_c + \Gamma_b \tag{36}$$

and

$$\frac{e_{15}^1}{G_L^1}u_1^+(\sigma)+\frac{e_{15}^2}{G_L^2}u_2^+(\sigma)-v_1^+(\sigma)-v_2^+(\sigma)$$

$$=\frac{e_{15}^1}{G_L^1}u_1^-(\sigma)+\frac{e_{15}^2}{G_L^2}u_2^-(\sigma)-v_1^-(\sigma)-v_2^-(\sigma) \quad \sigma\in\Gamma=\Gamma_c+\Gamma_b. \tag{37}$$

Let us define two functions $\omega_1(Z)$ and $\omega_2(Z)$, such that

$$\omega_1(Z)=u_1(Z)+u_2(Z)+\frac{e_{15}^1}{k_{11}^1}v_1(Z)+\frac{e_{15}^2}{k_{11}^2}v_2(Z) \tag{38}$$

$$\omega_2(Z)=\frac{e_{15}^1}{G_L^1}u_1(Z)+\frac{e_{15}^2}{G_L^2}u_2(Z)-v_1(Z)-v_2(Z) \tag{39}$$

which are holomorphic in the region $|Z|>0$. Thus, the functions $\omega_1(Z)$ and $\omega_2(Z)$ can be expanded in the region $|Z|>0$ in terms of Laurent series

$$\omega_1(Z)=\sum_{k=0}^{\infty}(e_kZ^{k+1}+f_kZ^{-(k+1)}) \tag{40}$$

$$\omega_2(Z)=\sum_{k=0}^{\infty}(g_kZ^{k+1}+h_kZ^{-(k+1)}). \tag{41}$$

Through lengthy manipulations (using eqns (28), (30), (34)–(41)), we obtain the following Riemann–Hilbert problems:

$$\begin{aligned} u_1^+(\sigma)-u_1^-(\sigma)&=0 \quad \sigma\in\Gamma_c\\ u_1^+(\sigma)+u_1^-(\sigma)&=F_1(\sigma) \quad \sigma\in\Gamma_b \end{aligned} \tag{42}$$

and

$$\begin{aligned} v_1^+(\sigma)-v_1^-(\sigma)&=0 \quad \sigma\in\Gamma_c\\ v_1^+(\sigma)+v_1^-(\sigma)&=F_2(\sigma) \quad \sigma\in\Gamma_b \end{aligned} \tag{43}$$

where

$$F_1(\sigma)=\sum_{k=0}^{\infty}(p_k\sigma^{k+1}+q_k\sigma^{-(k+1)}) \tag{44}$$

$$F_2(\sigma)=\sum_{k=0}^{\infty}(r_k\sigma^{k+1}+s_k\sigma^{-(k+1)}) \tag{45}$$

with

$$p_k=\frac{2}{\Delta_2(K_1K_4+K_2K_3)}[k_{11}^2(G_L^2K_4+e_{15}^2K_2)e_k+G_L^2(e_{15}^2K_4-k_{11}^2K_2)g_k] \tag{46}$$

$$q_k=\frac{2}{\Delta_2(K_1K_4+K_2K_3)}[k_{11}^2(G_L^2K_4+e_{15}^2K_2)f_k+G_L^2(e_{15}^2K_4-k_{11}^2K_2)h_k] \tag{47}$$

$$r_k = \frac{-2}{\Delta_2(K_1K_4+K_2K_3)}[k_{11}^2(G_L^2K_3-e_{15}^2K_1)e_k+G_L^2(e_{15}^2K_3+k_{11}^2K_1)g_k] \tag{48}$$

$$s_k = \frac{-2}{\Delta_2(K_1K_4+K_2K_3)}[k_{11}^2(G_L^2K_3-e_{15}^2K_1)f_k+G_L^2(e_{15}^2K_3+k_{11}^2K_1)h_k] \tag{49}$$

and

$$\Delta_2 = G_L^2k_{11}^2+(e_{15}^2)^2 \tag{50}$$

$$K_1 = \mu_1+(G_L^2k_{11}^2+\mu_1e_{15}^1e_{15}^2)/\Delta_2 \tag{51}$$

$$K_2 = G_L^2(e_{15}^2-\mu_2e_{15}^1)/\Delta_2 \tag{52}$$

$$K_3 = k_{11}^2(e_{15}^2-\mu_1e_{15}^1)/\Delta_2 \tag{53}$$

$$K_4 = \mu_2+(G_L^2k_{11}^2+\mu_2e_{15}^1e_{15}^2)/\Delta_2. \tag{54}$$

Since no singularities exist inside the inhomogeneity and the matrix, the complex potentials $\Psi_1(Z)$, $\Phi_1(Z)$, $\Psi_2(Z)$ and $\Phi_2(Z)$ must be holomorphic in their respective regions. Therefore, we can expand functions $\Psi_1(Z)$ and $\Phi_1(Z)$ into the following Laurent series:

$$\Psi_1(Z) = \sum_{k=0}^{\infty}(a_k^1Z^{k+1}+b_k^1Z^{-(k+1)}) \quad |Z|>1 \tag{55}$$

$$\Phi_1(Z) = \sum_{k=0}^{\infty}(c_k^1Z^{k+1}+d_k^1Z^{-(k+1)}) \quad |Z|>1 \tag{56}$$

while $\Psi_2(Z)$ and $\Phi_2(Z)$ into Taylor series:

$$\Psi_2(Z) = \sum_{k=0}^{\infty}a_k^2Z^{k+1} \quad |Z|<1 \tag{57}$$

$$\Phi_2(Z) = \sum_{k=0}^{\infty}c_k^2Z^{k+1} \quad |Z|<1 \tag{58}$$

where the constant terms relating to rigid-body displacements have been omitted. The coefficients $a_k^1$ and $c_k^1$ can be readily determined from the remote boundary conditions whereas the coefficients $b_k^1$, $d_k^1$, $a_k^2$ and $c_k^2$ are unknown and have to be determined as part of the solution.

Substituting (55)–(58) into (20)–(27), we have

$$u_1(Z) = \sum_{k=0}^{\infty}(k+1)(a_k^1Z^{k+1}-b_k^1Z^{-(k+1)}) \quad |Z|>1 \tag{59}$$

$$u_1(Z) = \sum_{k=0}^{\infty}(k+1)(\bar{b}_k^1Z^{k+1}-\bar{a}_k^1Z^{-(k+1)}) \quad 0<|Z|<1 \tag{60}$$

$$v_1(Z) = \sum_{k=0}^{\infty}(k+1)(c_k^1Z^{k+1}-d_k^1Z^{-(k+1)}) \quad |Z|>1 \tag{61}$$

$$v_1(Z) = \sum_{k=0}^{\infty} (k+1)(\overline{d_k^1} Z^{k+1} - \overline{c_k^1} Z^{-(k+1)}) \quad 0 < |Z| < 1 \tag{62}$$

$$u_2(Z) = \sum_{k=0}^{\infty} (k+1) a_k^2 Z^{k-1} \quad 0 < |Z| < 1 \tag{63}$$

$$u_2(Z) = -\sum_{k=0}^{\infty} (k+1) \overline{a_k^2} Z^{-(k+1)} \quad |Z| > 1 \tag{64}$$

$$v_2(Z) = \sum_{k=0}^{\infty} (k+1) c_k^2 Z^{k+1} \quad 0 < |Z| < 1 \tag{65}$$

$$v_2(Z) = -\sum_{k=0}^{\infty} (k+1) \overline{c_k^2} Z^{-(k+1)} \quad |Z| > 1. \tag{66}$$

The solution of (42) and (43) can be derived explicitly as

$$u_1(Z) = \frac{1}{2}\left\{\sum_{k=0}^{\infty} (p_k Z^{k+1} + q_k Z^{-(k-1)}) - X(Z)[U_\infty(Z) + U_0(Z)]\right\} + X(Z)R(Z) \tag{67}$$

$$v_1(Z) = \frac{1}{2}\left\{\sum_{k=0}^{\infty} (r_k Z^{k+1} + s_k Z^{-(k+1)}) - X(Z)[V_\infty(Z) + V_0(Z)]\right\} + X(Z)\tilde{R}(Z) \tag{68}$$

where $X(Z)$ is the Plemelj function of the problem defined by

$$X(Z) = [(Z-\sigma_1)(Z-\sigma_2)]^{-1/2}. \tag{69}$$

This function is holomorphic in the entire $Z$-plane, except along the cut of the bonded arc $\Gamma_b$ on which $X^+(\sigma) = -X^-(\sigma)$. The branch of $X(Z)$ with $\lim_{Z\to\infty}[ZX(Z)] = 1$ has been chosen so that it can be expanded at points $Z = \infty$ and $Z = 0$ such that

$$X(Z) = \sum_{n=0}^{\infty} \alpha_n^* Z^{-(n+1)} \quad \frac{1}{X(Z)} = \sum_{n=0}^{\infty} \alpha_n Z^{1-n} \quad \text{as } Z \to \infty \tag{70}$$

$$X(Z) = X(0)\sum_{n=0}^{\infty} \beta_n^* Z^n \quad \frac{1}{X(Z)} = \frac{1}{X(0)}\sum_{n=0}^{\infty} \beta_n Z^n \quad \text{as } Z \to 0 \tag{71}$$

where $X(0) = (\sigma_1\sigma_2)^{-1/2}$ and

$$\alpha_n^* = \sum_{m=0}^{n} \gamma_m^* \gamma_{n-m}^* \sigma_1^m \sigma_2^{n-m} \tag{72}$$

$$\beta_n^* = \sum_{m=0}^{n} \gamma_m^* \gamma_{n-m}^* \sigma_1^{-m} \sigma_2^{-(n-m)} \tag{73}$$

$$\alpha_n = \sum_{m=0}^{n} \gamma_m \gamma_{n-m} \sigma_1^m \sigma_2^{n-m} \tag{74}$$

$$\beta_n = \sum_{m=0}^{n} \gamma_m \gamma_{n-m} \sigma_1^{-m} \sigma_2^{-(n-m)} \tag{75}$$

with

$$\gamma_m = \begin{cases} 1 & m = 0 \\ -\dfrac{\gamma_{m-1}^*}{2m} & m \geqslant 1 \end{cases} \quad \gamma_m^* = \frac{(2m)!}{2^{2m}(m!)^2}. \tag{76}$$

The functions $U_\infty(Z)$, $U_0(Z)$, $V_\infty(Z)$, $V_0(Z)$, $R(Z)$ and $\tilde{R}(Z)$ in (67) and (68) are given by

$$U_\infty(Z) = \sum_{n=0}^{\infty} p_n \sum_{m=0}^{n+1} \alpha_m Z^{n+2-m} \tag{77}$$

$$U_0(Z) = \frac{1}{X(0)} \sum_{n=0}^{\infty} q_n \sum_{m=0}^{n+1} \beta_m Z^{m-n-1} \tag{78}$$

$$V_\infty(Z) = \sum_{n=0}^{\infty} r_n \sum_{m=0}^{n+1} \alpha_m Z^{n+2-m} \tag{79}$$

$$V_0(Z) = \frac{1}{X(0)} \sum_{n=0}^{\infty} s_n \sum_{m=0}^{n+1} \beta_m Z^{m-n-1} \tag{80}$$

$$R(Z) = h^* + \sum_{k=0}^{\infty} l_k Z^{k+1} + \sum_{k=0}^{\infty} h_k Z^{-(k+1)} \tag{81}$$

$$\tilde{R}(Z) = \tilde{h}^* + \sum_{k=0}^{\infty} \tilde{l}_k Z^{k+1} + \sum_{k=0}^{\infty} \tilde{h}_k Z^{-(k+1)} \tag{82}$$

with

$$h^* = -\frac{1}{X(0)} \sum_{m=0}^{\infty} (m+1)\beta_{m+1} \bar{a}_m^1 \tag{83}$$

$$h_k = -\frac{1}{X(0)} \sum_{m=0}^{\infty} (m+k+1)\beta_m \bar{a}_{m+k}^1 \quad (k \geqslant 0) \tag{84}$$

$$l_0 = \sum_{m=0}^{\infty} (m+1)\alpha_{m+1} a_m^1 \tag{85}$$

$$l_k = \sum_{m=0}^{\infty} (m+k)\alpha_m a_{m+k-1}^1 \quad (k \geqslant 1) \tag{86}$$

$$\tilde{h}^* = -\frac{1}{X(0)} \sum_{m=0}^{\infty} (m+1)\beta_{m+1} \bar{c}_m^1 \tag{87}$$

$$\tilde{h}_k = -\frac{1}{X(0)} \sum_{m=0}^{\infty} (m+k+1)\beta_m \bar{c}_{m+k}^1 \quad (k \geqslant 0) \tag{88}$$

$$\tilde{l}_0 = \sum_{m=0}^{\infty} (m+1)\alpha_{m+1} c_m^1 \tag{89}$$

$$\tilde{l}_k = \sum_{m=0}^{\infty} (m+k)\alpha_m c_{m+k-1}^1 \quad (k \geqslant 1). \tag{90}$$

From (38) and (39), we have

$$u_2(Z) = \frac{G_L^2}{\Delta_2}\left\{k_{11}^2\left[\omega_1(Z) - u_1(Z) - \frac{e_{15}^1}{k_{11}^1} v_1(Z)\right] + e_{15}^2\left[\omega_2(Z) - \frac{e_{15}^1}{G_L^1} u_1(Z) + v_1(Z)\right]\right\} \tag{91}$$

$$v_2(Z) = \frac{k_{11}^2}{\Delta_2}\left\{e_{15}^2\left[\omega_1(Z) - u_1(Z) - \frac{e_{15}^1}{k_{11}^1} v_1(Z)\right] - G_L^2\left[\omega_2(Z) - \frac{e_{15}^1}{G_L^1} u_1(Z) + v_1(Z)\right]\right\}. \tag{92}$$

Equation (67), (68), (91) and (92) constitute the general explicit form of the solution to the present problem with the unknown coefficients $e_k$, $f_k$, $g_k$ and $h_k$ (contained in (40) and (41)) to be determined.

Substituting (59)–(66) into (38) and (39) and comparing the like powers to $Z$ with those in (40) and (41), we obtain

$$e_k = (k+1)\left(a_k^1 + \frac{e_{15}^1}{k_{11}^1} c_k^1\right) \tag{93}$$

$$f_k = -(k+1)\left(b_k^1 + \overline{a_k^2} + \frac{e_{15}^1}{k_{11}^1} d_k^1 + \frac{e_{15}^2}{k_{11}^2}\overline{c_k^2}\right) \tag{94}$$

$$f_k = -\bar{e}_k \tag{95}$$

$$g_k = (k+1)\left(\frac{e_{15}^1}{G_L^1} a_k^1 - c_k^1\right) \tag{96}$$

$$h_k = -(k+1)\left(\frac{e_{15}^1}{G_L^1} b_k^1 + \frac{e_{15}^2}{G_L^2}\overline{a_k^2} - d_k^1 - \overline{c_k^2}\right) \tag{97}$$

$$h_k = -\bar{g}_k. \tag{98}$$

Expanding (67) and (68) into Laurent series' in the matrix and comparing them with (59) and (61) render that

$$b_0^1 = -\frac{1}{2}\left[q_0 - \sum_{p=0}^{\infty} p_p \sum_{m=0}^{p+1} (\alpha_m \alpha_{p+2-m}^*) - \frac{1}{X(0)}\sum_{p=0}^{\infty} (q_p \beta_{p+1} \alpha_0^*)\right] - \alpha_0^* h^* - \sum_{m=0}^{\infty} \alpha_{m+1}^* l_m \tag{99}$$

$$d_0^1 = -\frac{1}{2}\left[s_0 - \sum_{p=0}^{\infty} r_p \sum_{m=0}^{p+1} (\alpha_m \alpha_{p+2-m}^*) - \frac{1}{X(0)}\sum_{p=0}^{\infty} (s_p \beta_{p+1} \alpha_0^*)\right] - \alpha_0^* \tilde{h}^* - \sum_{m=0}^{\infty} \alpha_{m+1}^* \tilde{l}_m \tag{100}$$

$$(k+1)b_k^1 = -\frac{1}{2}\left[q_k - \sum_{p=0}^{\infty} p_p \sum_{m=0}^{p+1} (\alpha_m \alpha^*_{p+k+2-m}) - \frac{1}{X(0)} \sum_{p=0}^{\infty} q_p \sum_{m=0}^{k,p+1} (\beta_{p+1-m} \alpha^*_{k-m})\right] - \alpha_k^* h^* - \sum_{m=0}^{\infty} \alpha^*_{k-m+1} l_m - \sum_{m=0}^{k-1} \alpha_m^* h_{k-m-1} \quad (k \geqslant 1) \tag{101}$$

$$(k+1)d_k^1 = -\frac{1}{2}\left[s_k - \sum_{m=0}^{\infty} r_p \sum_{m=0}^{p-1} (\alpha_m \alpha^*_{p+k+2-m}) - \frac{1}{X(0)} \sum_{p=0}^{\infty} s_p \sum_{m=0}^{k,p+1} (\beta_{p+1-m} \alpha^*_{k-m})\right] - \alpha_k^* \tilde{h}^* - \sum_{m=0}^{\infty} \alpha^*_{k+m+1} \tilde{l}_m - \sum_{m=0}^{k-1} \alpha_m^* \tilde{h}_{k-m-1} \quad (k \geqslant 1) \tag{102}$$

where a smaller value of the summation upper limit ($k$ or $p+1$) is implied.

The unknown coefficients $e_k, f_k, g_k, h_k, a_k^2, b_k^1, d_k^1$ can be obtained by solving eqns (93)–(102), once $a_k^1$ and $c_k^1$ are determined from the boundary conditions at infinity. This allows the determination of the stress and electric fields in the inhomogeneity and the matrix.

An interesting aspect of the present solution is the ability to determine the energy release rate of the resulting curvilinear crack. The internal energy $W$ in a piezoelectric solid of a unit thickness and a cross-sectional area $D$ can be written as

$$W = \frac{1}{2}\iint_D (\sigma_{zx}\gamma_{zx} + \sigma_{zy}\gamma_{zy} + D_x E_x + D_y E_y)\,\mathrm{d}s. \tag{103}$$

It can be shown that

$$W = W_0 + \Delta W \tag{104}$$

where $W_0$ is the internal energy in the absence of inhomogeneities and $\Delta W$ represents the change of the internal energy due to the presence of the inhomogeneity and its consequent debonding. Following the approach of Sih and Liebowitz (1968) and considering the interfacial conditions of the debonding of an inhomogeneity, $\Delta W$ can be evaluated as

$$\Delta W = \frac{\pi}{2G_L^1}(a_0^1 b_0^1 + \overline{a_0^1 b_0^1}) + \frac{\pi}{2k_{11}^1}(c_0^1 d_0^1 + \overline{c_0^1 d_0^1}). \tag{105}$$

Therefore, the energy release rate can be defined as (Zhang and Tong, 1996):

$$g_1 = -\frac{1}{a}\frac{\partial(\Delta W)}{\partial \beta_1} \tag{106}$$

for crack tip $t_1$, and

$$g_2 = \frac{1}{a}\frac{\partial(\Delta W)}{\partial \beta_2} \tag{107}$$

for crack tip $t_2$, see Fig. 1(a).

The previous general solution is valid for any antiplane mechanical loads and inplane electric fields at infinity. Let us now consider the case of an arc-crack symmetrically placed

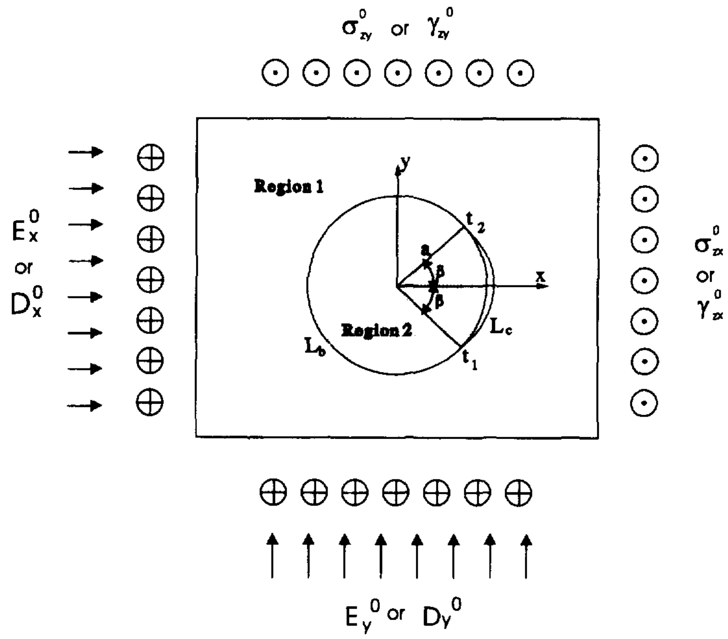



Fig. 2. A circular arc-crack at the interface of two bonded materials.

with respect to the $x$-axis ($\beta_2 = -\beta_1 = \beta$), Fig. 2, under remote uniform loading conditions. By applying the remote loading conditions, we can determine

$$a_0^1 = a(P_0^\bullet + iP_0^*) \tag{108}$$

$$c_0^1 = a(Q_0^\bullet + iQ_0^*) \tag{109}$$

$$a_k^1 = c_k^1 = 0 \quad (k \geqslant 1) \tag{110}$$

where $P_0^\bullet$, $P_0^*$, $Q_0^\bullet$ and $Q_0^*$ are evaluated for four special cases in the Appendix.

Accordingly, the energy release rates for the symmetric arc-crack can be derived such that

$$\begin{aligned} g_1 = {} & \frac{\pi a \sin\beta}{4G_L^1}[P_0^\bullet(2P_0^\bullet - p_0^\bullet)(1+\cos\beta) + P_0^*(2P_0^* - p_0^*)(1-\cos\beta)] \\ & + \frac{\pi a \sin^2\beta}{4G_L^1}[P_0^\bullet(2P_0^* - p_0^*) + P_0^*(2P_0^\bullet - p_0^\bullet)] \\ & + \frac{\pi a \sin\beta}{4k_{11}^1}[Q_0^\bullet(2Q_0^\bullet - r_0^\bullet)(1+\cos\beta) + Q_0^*(2Q_0^* - r_0^*)(1-\cos\beta)] \\ & + \frac{\pi a \sin^2\beta}{4k_{11}^1}[Q_0^\bullet(2Q_0^* - r_0^*) + Q_0^*(2Q_0^\bullet - r_0^\bullet)] \end{aligned} \tag{111}$$

for crack tip $t_1$, and

$$\begin{aligned} g_2 = {} & \frac{\pi a \sin\beta}{4G_L^1}[P_0^\bullet(2P_0^\bullet - p_0^\bullet)(1+\cos\beta) + P_0^*(2P_0^* - p_0^*)(1-\cos\beta)] \\ & - \frac{\pi a \sin^2\beta}{4G_L^1}[P_0^\bullet(2P_0^* - p_0^*) + P_0^*(2P_0^\bullet - p_0^\bullet)] \end{aligned}$$

$$+\frac{\pi a\sin\beta}{4k_{11}^{1}}[Q_0^{\bullet}(2Q_0^{\bullet}-r_0^{\bullet})(1+\cos\beta)+Q_0^{*}(2Q_0^{*}-r_0^{*})(1-\cos\beta)]$$

$$-\frac{\pi a\sin^2\beta}{4k_{11}^{1}}[Q_0^{\bullet}(2Q_0^{*}-r_0^{*})+Q_0^{*}(2Q_0^{\bullet}-r_0^{\bullet})] \tag{112}$$

for crack tip $t_2$, where

$$p_0^{\bullet}=\frac{2}{\Delta_2(K_1K_4+K_2K_3)}[k_{11}^2(G_L^2K_4+e_{15}^2K_2)e_0^{\bullet}+G_L^2(e_{15}^2K_4-k_{11}^2K_2)g_0^{\bullet}] \tag{113}$$

$$p_0^{*}=\frac{2}{\Delta_2(K_1K_4+K_2K_3)}[k_{11}^2(G_L^2K_4+e_{15}^2K_2)e_0^{*}+G_L^2(e_{15}^2K_4-k_{11}^2K_2)g_0^{*}] \tag{114}$$

$$r_0^{\bullet}=\frac{-2}{\Delta_2(K_1K_4+K_2K_3)}[k_{11}^2(G_L^2K_3-e_{15}^2K_1)e_0^{\bullet}+G_L^2(e_{15}^2K_3+k_{11}^2K_1)g_0^{\bullet}] \tag{115}$$

$$r_0^{*}=\frac{-2}{\Delta_2(K_1K_4+K_2K_3)}[k_{11}^2(G_L^2K_3-e_{15}^2K_1)e_0^{*}+G_L^2(e_{15}^2K_3+k_{11}^2K_1)g_0^{*}] \tag{116}$$

with $\Delta_2$, $K_1$, $K_2$, $K_3$, $K_4$ given by (50)–(54) and

$$e_0^{\bullet}=P_0^{\bullet}+\frac{e_{15}^1}{k_{11}^1}Q_0^{\bullet} \tag{117}$$

$$e_0^{*}=P_0^{*}+\frac{e_{15}^1}{k_{11}^1}Q_0^{*} \tag{118}$$

$$g_0^{\bullet}=\frac{e_{15}^1}{G_L^1}P_0^{\bullet}-Q_0^{\bullet} \tag{119}$$

$$g_0^{*}=\frac{e_{15}^1}{G_L^1}P_0^{*}-Q_0^{*}. \tag{120}$$

## 4. EXAMPLES AND DISCUSSION

In the following subsections, we examine a number of interesting cases which demonstrate the validity and versatility of the general formulations.

### 4.1. *Straight crack in a homogeneous piezoelectric material*

Consider a straight crack with length $2l$ along the $y$-axis in a homogeneous material under antiplane shear and in plane electric field (Fig. 3). This problem has been studied by many researchers (see, for example, Pak, 1990; Zhang and Tong, 1996). In this case, we have

$$G_L^1=G_L^2=G_L \quad k_{11}^1=k_{11}^2=k_{11} \quad e_{15}^1=e_{15}^2=e_{15} \tag{121}$$

$$p_0^{\bullet}=P_0^{\bullet} \quad p_0^{*}=P_0^{*} \quad r_0^{\bullet}=Q_0^{\bullet} \quad r_0^{*}=Q_0^{*}. \tag{122}$$

The solution to this special case can be easily deduced from our general formulations by letting the radius $a$ of the circular inhomogeneity tend to infinity and the angles $\beta_1$ and $\beta_2$

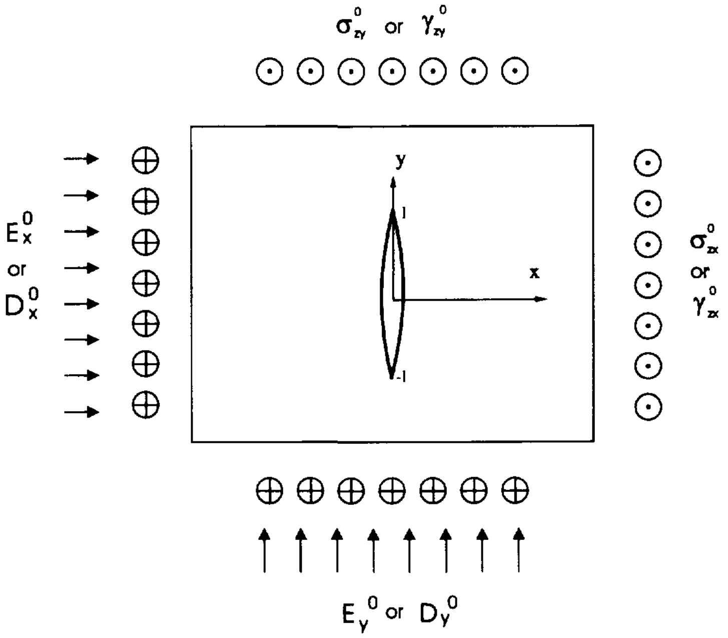



Fig. 3. A straight crack in a homogeneous piezoelectric material.

approach zero such that $a\beta_2 = -a\beta_1 = l$. For example, the energy release rate can be obtained from (111) and (112) as being

$$g_1 = g_2 = \frac{\pi l}{2G_L}(P_0^\bullet)^2 + \frac{\pi l}{2k_{11}}(Q_0^\bullet)^2. \tag{123}$$

Expression (123) confirms the previous result of Zhang and Tong (1996).

4.2. *Straight crack along the interface of two dissimilar piezoelectrical materials*

Consider the problem of a straight crack lying along the $y$-axis from $-l$ to $+l$ between two dissimilar materials, as depicted in Fig. 4. In this case, the radius of the circular

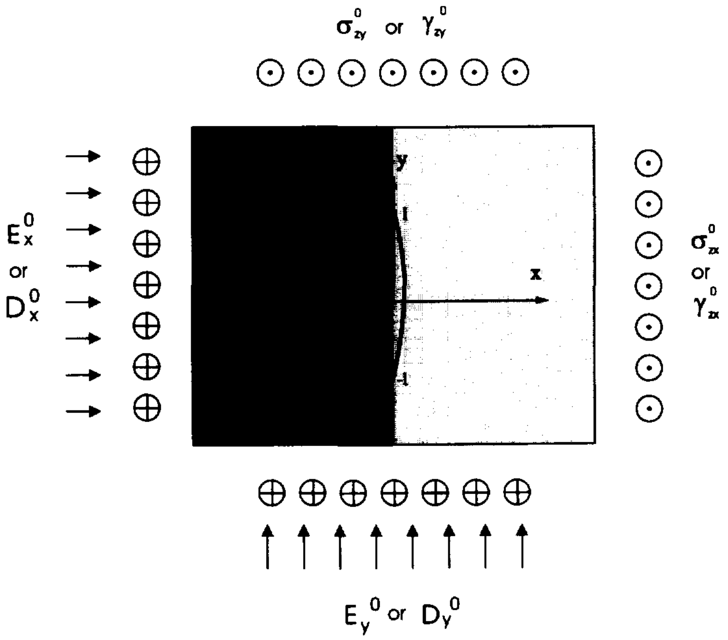



Fig. 4. A straight crack along the interface of two dissimilar piezoelectric materials.

inhomogeneity, $a$, tends to infinity and the angles $\beta_1$ and $\beta_2$ approach zero such that $a\beta_2 = -a\beta_1 = l$. Accordingly, the energy rate can be obtained from (128) as being

$$g_1 = g_2 = \frac{\pi l}{2G_L^1} P_0^{\bullet}(2P_0^{\bullet} - p_0^{\bullet}) + \frac{\pi l}{2k_{11}^1} Q_0^{\bullet}(2Q_0^{\bullet} - r_0^{\bullet}) \tag{124}$$

where $p_0^{\bullet}$ and $r_0^{\bullet}$ are given by (113) and (115).

4.3. *Circular arc-crack in a homogeneous piezoelectric material*

Consider a circular arc-crack in an infinite piezoelectric matrix subject to an inplane electric field as well as an antiplane shear at infinity, as shown in Fig. 5. Without loss of generality, the centre of the crack $L_c$ can be assumed to lie on the positive $x$-axis and the central angle subtended by $L_c$ is assumed to be $2\beta$. The solution can be easily deduced from our general solution by considering $G_L^1 = G_L^2 = G_L$, $k_{11}^1 = k_{11}^2 = k_{11}$, $e_{15}^1 = e_{15}^2 = e_{15}$. In this case, the energy release rates can be derived as follows:

$$g_1 = \frac{\pi a \sin\beta}{4G_L}[(P_0^{\bullet})^2(1+\cos\beta) + (P_0^*)^2(1-\cos\beta) + 2P_0^{\bullet}P_0^*\sin\beta]$$
$$+ \frac{\pi a \sin\beta}{4k_{11}}[(Q_0^{\bullet})^2(1+\cos\beta) + (Q_0^*)^2(1-\cos\beta) + 2Q_0^{\bullet}Q_0^*\sin\beta] \tag{125}$$

for crack tip $t_1$, and

$$g_2 = \frac{\pi a \sin\beta}{4G_L}[(P_0^{\bullet})^2(1+\cos\beta) + (P_0^*)^2(1-\cos\beta) - 2P_0^{\bullet}P_0^*\sin\beta]$$
$$+ \frac{\pi a \sin\beta}{4k_{11}}[(Q_0^{\bullet})^2(1+\cos\beta) + (Q_0^*)^2(1-\cos\beta) - 2Q_0^{\bullet}Q_0^*\sin\beta] \tag{126}$$

for crack tip $t_2$.

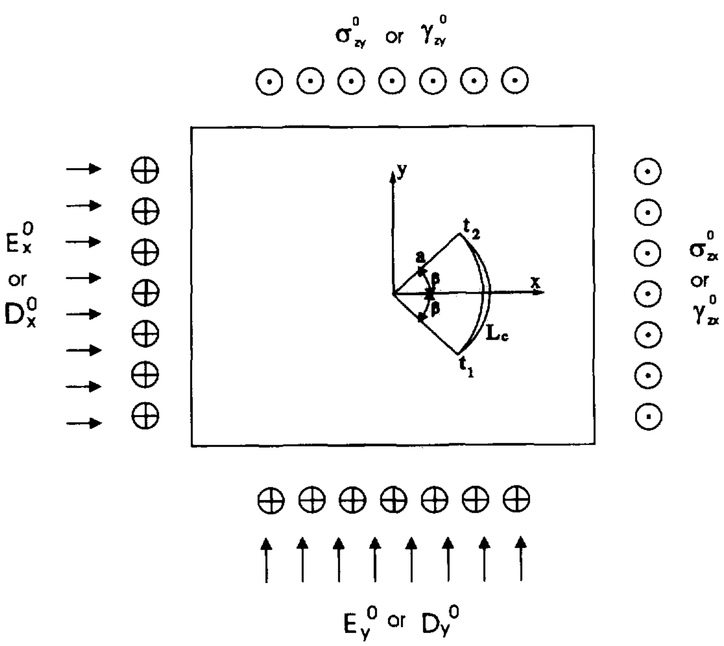



Fig. 5. A circular arc-crack in a homogeneous piezoelectric material.

4.4. *Partially-debonded circular piezoelectric inhomogeneity embedded in an elastic matrix*

Consider a partially debonded circular piezoelectric inhomogeneity in an infinite elastic matrix ($e_{15}^1 = 0$ and $k_{11}^1 = 0$), see Fig. 2. Piezoelectric sensors are often made in this configuration, where a piezoelectric rod is embedded in an elastic material (usually polymers). Let us suppose that the material is subjected to uniform stresses $\sigma_{zx}^0$ and $\sigma_{zy}^0$ at infinity. In this case, we have

$$p_0^{\bullet} = \frac{2\sigma_{zx}^0}{1+\mu_1(1+m)} \tag{127}$$

$$p_0^{*} = \frac{-2\sigma_{zy}^0}{1+\mu_1(1+m)} \tag{128}$$

$$r_0^{\bullet} = r_0^{*} = 0 \tag{129}$$

where

$$m = \frac{(e_{15}^2)^2}{G_L^2 k_{11}^2}. \tag{130}$$

The energy release rate is obtained by substituting (127)–(129) into (111) and (112), as follows:

$$g_1 = \frac{\pi a \mu_1 (1+m)\sin\beta}{2G_L^1[1+\mu_1(1+m)]}[(1+\cos\beta)(\sigma_{zx}^0)^2+(1-\cos\beta)(\sigma_{zy}^0)^2-2\sigma_{zx}^0\sigma_{zy}^0\sin\beta] \tag{131}$$

for crack tip $t_1$ and

$$g_2 = \frac{\pi a \mu_1 (1+m)\sin\beta}{2G_L^1[1+\mu_1(1+m)]}[(1+\cos\beta)(\sigma_{zx}^0)^2+(1-\cos\beta)(\sigma_{zy}^0)^2+2\sigma_{zx}^0\sigma_{zy}^0\sin\beta] \tag{132}$$

for crack tip $t_2$.

The variation of the normalized energy release rates $G_1(=g_1/(\pi a/2G_L^1)(\sigma^0)^2)$ and $G_2(=g_2/(\pi a/2G_L^1)(\sigma^0)^2)$ vs the crack angle $\beta$ is shown in Fig. 6 for the case where

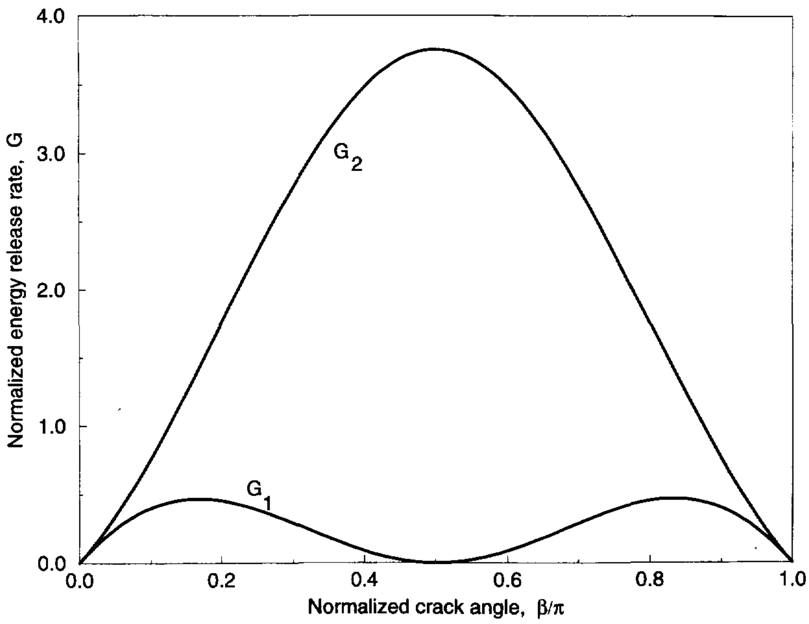



Fig. 6. Variation of normalized internal energy release rates $G_1$ and $G_2$ vs crack angle $\beta$ for the case where $\sigma_{zx}^0 = \sigma_{zy}^0 = \sigma^0$, and $\mu_1 = 10$.

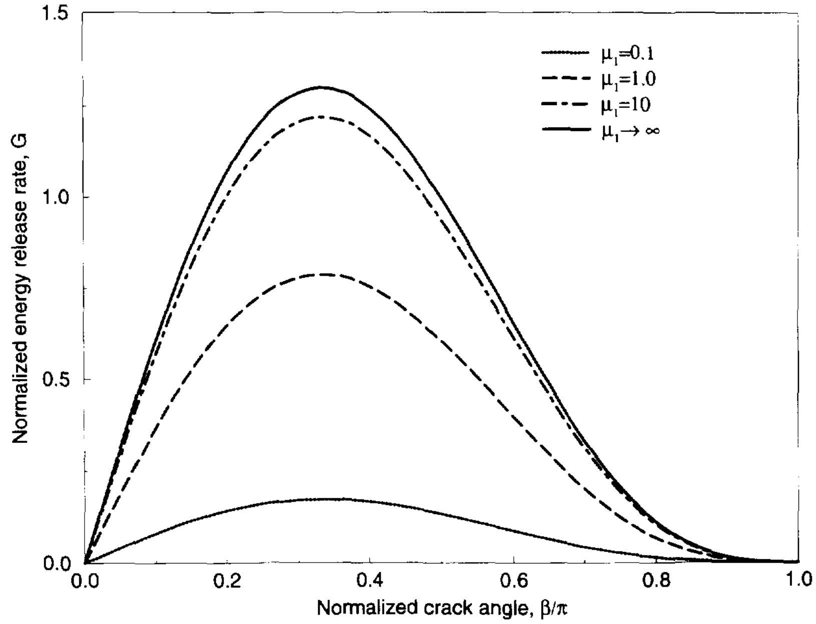



Fig. 7. Variation of normalized internal energy release rate $G$ vs crack angle $\beta$ for some specific values of $\mu_1$ when $\sigma^0_{zx} = \sigma^0$ and $\sigma^0_{zy} = 0$.

$\sigma^0_{zx} = \sigma^0_{zy} = \sigma^0$. The material properties of piezoelectric sensor (e.g. PZT-5H) are taken as follows (Par, 1990) : $G^2_L = 3.53 \times 10^{10}$ N/m$^2$, $k^2_{11} = 1.51 \times 10^{-8}$ C/Vm, $e^2_{15} = 17.0$ C/m$^2$ with the longitudinal shear modulus of the elastic matrix $G^1_L = G^2_L/10$ ($\mu_1 = 10$). It is observed from Fig. 6 that $G_2$ is greater than $G_1$ for any crack angle $\beta$ for the case where $\sigma^0_{zx} = \sigma^0_{zy} = \sigma^0$ ; which means that the crack will first propagate at point $t_2$. It can be also found that $G_2$ reaches its maximum value at $\beta = \pi/2$ and $G_1$ reaches its maximum at $\beta = \pi/6$ and $\beta = 5\pi/6$. This trend is reversed for the case where $\sigma^0_{zx} = -\sigma^0_{zy} = \sigma^0$ (not shown).

Figure 7 depicts the change of the normalized energy release rate $G$ vs crack angle $\beta$ for some specific values of $\mu_1 (= G^2_L/G^1_L)$ for the case where $\sigma^0_{zx} = \sigma^0$, $\sigma_{zy} = 0$, while Fig. 8 shows the variation of $G$ for the case where $\sigma^0_{zx} = 0$, $\sigma^0_{zy} = \sigma^0$. In both cases, we have $G_1 = G_2 = G$. This implies that under critical loading conditions, the crack will propagate simultaneously at points $t_1$ and $t_2$. It is seen from Fig. 7 that $G$ increases as $\mu_1$ increases.

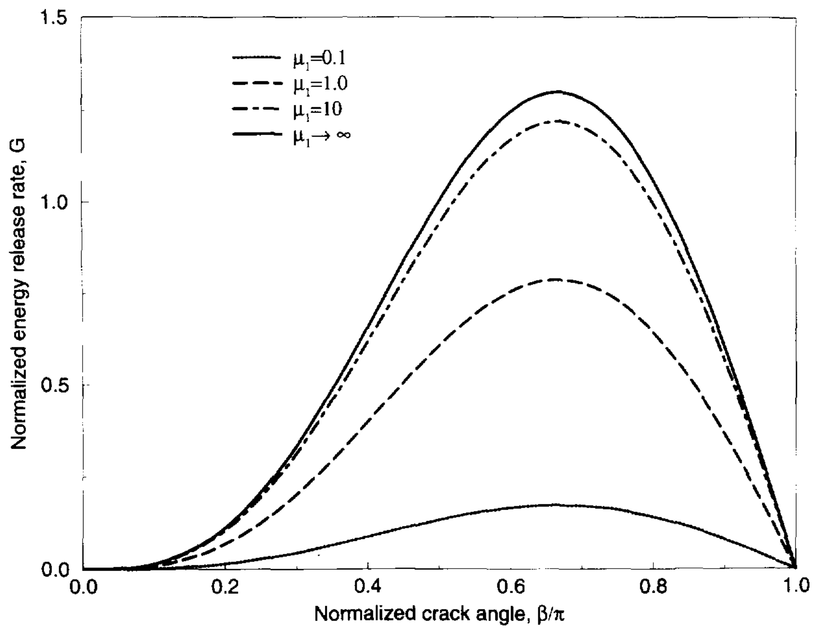



Fig. 8. Variation of normalized internal energy release rate $G$ vs crack angle $\beta$ for some specific values of $\mu_1$ when $\sigma^0_{zx} = 0$ and $\sigma^0_{zy} = \sigma^0$.

For a given $\mu_1$, $G$ varies with respect to $\beta$ and reaches its maximum value at $\beta = \pi/3$. Figure 8 shows a mirror image of the trend observed in Fig. 7. In this case, the maximum value of $G$ is reached at $\beta = 2\pi/3$.

## 5. CONCLUSION

A generalized and mathematically rigorous model is developed to treat the partially-debonded circular inhomogeneity problem in piezoelectric materials under antiplane shear and inplane electric field using the complex variable method. The principle of analytical continuation and the complex series expansion method were employed to reduce the formulations into two Riemann–Hilbert problems. This enabled the explicit determination of the complex potentials inside the inhomogeneity and the matrix. The resulting closed form expressions were then used to obtain the energy release rate. The validity and versatility of the current generalized solution have been demonstrated by application to some particular examples and the influence of crack angle and material properties upon the energy release rate is discussed.

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## APPENDIX

*The determination of the coefficients $P_0^{\bullet}$, $P_0^*$, $Q_0^{\bullet}$ and $Q_0^*$*

Case 1: The matrix is subjected to uniform strains $\gamma_{zx}^0$ and $\gamma_{zy}^0$ as well as uniform electric fields $E_x^0$ and $E_y^0$ at infinity. In this case, we can obtain

$$P_0^{\bullet} = G_L^1 \gamma_{zx}^0 \tag{A.1}$$

$$P_0^* = -G_L^1 \gamma_{zy}^0 \tag{A.2}$$

$$Q_0^{\bullet} = -k_{11}^1 E_x^0 \tag{A.3}$$

$$Q_0^* = k_{11}^1 E_y^0. \tag{A.4}$$

Case 2: The matrix is subjected to uniform stresses $\sigma_{zx}^0$ and $\sigma_{zy}^0$ as well as uniform electric displacements $D_x^0$ and $D_y^0$ at infinity. As a result, we can deduce the following expressions:

$$P_0^\bullet = \frac{G_L^1(k_{11}^1\sigma_{zx}^0 + e_{15}^1 D_x^0)}{G_L^1 k_{11}^1 + (e_{15}^1)^2} \tag{A.5}$$

$$P_0^* = -\frac{G_L^1(k_{11}^1\sigma_{zy}^0 + e_{15}^1 D_y^0)}{G_L^1 k_{11}^1 + (e_{15}^1)^2} \tag{A.6}$$

$$Q_0^\bullet = \frac{k_{11}^1(e_{15}^1\sigma_{zx}^0 - G_L^1 D_x^0)}{G_L^1 k_{11}^1 + (e_{15}^1)^2} \tag{A.7}$$

$$Q_0^* = -\frac{k_{11}^1(e_{15}^1\sigma_{zy}^0 - G_L^1 D_y^0)}{G_L^1 k_{11}^1 + (e_{15}^1)^2}. \tag{A.8}$$

Case 3: The matrix is subjected to uniform strains $\gamma_{zx}^0$ and $\gamma_{zy}^0$ as well as uniform electric displacements $D_x^0$ and $D_y^0$ at infinity. This leads to

$$P_0^\bullet = G_L^1\gamma_{zx}^0 \tag{A.9}$$

$$P_0^* = -G_L^1\gamma_{zy}^0 \tag{A.10}$$

$$Q_0^\bullet = e_{15}^1\gamma_{zx}^0 - D_x^0 \tag{A.11}$$

$$Q_0^* = -e_{15}^1\gamma_{zy}^0 + D_y^0. \tag{A.12}$$

Case 4: The matrix is subjected to uniform stresses $\sigma_{zx}^0$, $\sigma_{zy}^0$ as well as uniform electric fields $E_x^0$ and $E_y^0$ at infinity. In this case, we can find that

$$P_0^\bullet = \sigma_{zx}^0 + e_{15}^1 E_x^0 \tag{A.13}$$

$$P_0^* = -\sigma_{zy}^0 - e_{15}^1 E_y^0 \tag{A.14}$$

$$Q_0^\bullet = -k_{11}^1 E_x^0 \tag{A.15}$$

$$Q_0^* = k_{11}^1 E_y^0. \tag{A.16}$$